SOME PROBLEMS OF THE THEORY OF DYNAMIC PROGRAMMING FOR NONLINEAR SYSTEMS

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This paper deals with certain questions of the theory of dynamic programming [1] related to the realization of a chosen strategy of control of motion. It contains the problem of selection of controlling forces which assure the realization of a state of motion prescribed in the phase space (or subspace) of the controlled nonlinear system, or which assure that the nonlinear system passes through predicted states at certain determined instants of time. A similar problem for linear systems was considered in a previous paper of the author [2].

1. The equations of motion of a system with continuous processes can be written in the following form

$$\sum_{k=1}^{n} f_{jk}(D)\, y_k = x_j(t) + q_j(t) +$$

$$+ \psi_j(y_1, \dot y_1, \ldots, y_1^{(m_1-1)}, \ldots, y_n, \dot y_n, \ldots, y_n^{(m_n-1)}, t) \qquad (j=1,\ldots,n) \qquad (1.1)$$

Here, y_k denote the generalized coordinates, $x_j(t)$ are given external forces, and $q_i(t)$ are additional external forces whose variation in time should be selected in such a way that a prescribed motion be performed. The symbols $f_{jk}(D)$ denote polynomials of D whose coefficients are given functions of time; $D = d/dt$ is the operator of differentiation with respect to time. The highest order of D in the polynomials $f_{jk}(D)$ $(j = 1, \ldots, n)$ is denoted by m_k for each k $(k = 1, \ldots, n)$, i.e. m_k is the order of the highest derivative of y_k with respect to time which appears on the left-hand sides of Equations (1.1).

The functions ψ_j $(j = 1, \ldots, n)$ on the right-hand sides of Equations (1.1) are certain nonlinear functions of the generalized coordinates. We shall assume that all these functions are continuous in all their variables in a closed region, and that in this region they satisfy the Lipschitz conditions in the variables

$$y_1, \ \dot{y}_1, \ldots, y_1^{(m_1-1)}, \ldots, y_n, \qquad \dot{y}_n, \ldots, y_n^{(m_n-1)}$$

We note that Equations (1.1) apply also to systems which contain the effects of usually applicable forces, which are functions of incompatibility. The necessary external forces are included in the given external forces $x_j(t)$, while the forces being functions of the controlled coordinates and their derivatives are taken into account in the left-hand sides of Equations (1.1) and also in the nonlinear functions ψ_j.

The system of Equations (1.1) can be rewritten in the form

$$\begin{aligned} &b_{j1}(t)\, y_1^{(m_1)} + b_{j2}(t)\, y_2^{(m_2)} + \ldots + b_{jn}(t)\, y_n^{(m_n)} = \\ &= S_j(y_1, \dot{y}_1, \ldots, y_1^{(m_1-1)}, \ldots, y_n, \dot{y}_n, \ldots, y_n^{(m_n-1)}) + x_j(t) + q_j(t) + \\ &+ \psi_j(y_1, \dot{y}_1, \ldots, y_1^{(m_1-1)}, \ldots, y_n, \dot{y}_n, \ldots, y_n^{(m_n-1)}, t) \qquad (j=1, \ldots, n) \end{aligned} \tag{1.2}$$

where the functions S_j are linear functions of their variables.

Assuming that the determinant

$$\Delta^* = |\, b_{jk}(t)\,| \tag{1.3}$$

is not identically equal to zero, we obtain from (1.2)

$$\begin{aligned} y_j^{(m_j)} = \Phi_j(y_1, \dot{y}_1, \ldots, y_1^{(m_1-1)}, \ldots, y_n, \dot{y}_n, \ldots, y_n^{(m_n-1)}) + \\ + \sum_{k=1}^{n} \frac{B_{kj}(t)}{\Delta^*(t)} [x_k(t) + q_k(t) + \\ + \psi_k(y_1, \dot{y}_1, \ldots, y_1^{(m_1-1)}, \ldots, y_n, \dot{y}_n, \ldots, y_n^{(m_n-1)}, t)] \qquad (j=1, \ldots, n) \end{aligned} \tag{1.4}$$

Here Φ_j are certain nonlinear functions, and B_{ij} are the algebraic complements of the elements b_{ij} in the determinant (1.3).

We introduce now new variables z_i by means of the relations

$$z_1 = y_1, \ z_2 = \dot{y}_1, \ldots, z_{m_1} = y_1^{(m_1-1)}, \ldots, z_r = y_n^{(m_n-1)} \tag{1.5}$$

where

$$r = m_1 + m_2 + \ldots + m_n \tag{1.6}$$

The linear combinations of the external forces $x_k(t)$, $q_k(t)$, and the functions ψ_k, which appear on the right-hand sides of Equations (1.4), will be denoted in the following way

$$X_{\sigma_j}(t) = \sum_{k=1}^{n} \frac{B_{kj}(t)}{\Delta^*(t)} x_k(t), \qquad Q_{\sigma_j}(t) = \sum_{k=1}^{n} \frac{B_{kj}(t)}{\Delta^*(t)} q_k(t)$$

$$\Psi_{\sigma_j}(z_1, \ldots, z_r, t) = \sum_{k=1}^{n} \frac{B_{kj}(t)}{\Delta^*(t)} \psi_k(z_1, \ldots, z_r, t) \quad (\sigma_j = \sigma_1, \ldots, \sigma_n) \tag{1.7}$$

where

$$\sigma_1 = m_1, \qquad \sigma_2 = m_1 + m_2, \ldots, \sigma_n = r \tag{1.8}$$

Equations (1.4) can now be rewritten as follows

$$\begin{gathered} \dot{z}_1 - z_2 = 0 \\ \cdots\cdots \\ \dot{z}_{m_1} - \Phi_1(z_1, \ldots, z_r) = X_{\sigma_1}(t) + Q_{\sigma_1}(t) + \Psi_{\sigma_1}(z_1, \ldots, z_r, t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{z}_r - \Phi_n(z_1, \ldots, z_r) = X_{\sigma_n}(t) + Q_{\sigma_n}(t) + \Psi_{\sigma_n}(z_1, \ldots, z_r, t) \end{gathered} \tag{1.9}$$

Because of the linearity of the functions $\Phi_j(z_1, \ldots, z_r)$, Equations (1.9) can be represented in the form

$$\dot{z}_j + \sum_{k=1}^{r} a_{jk}(t) z_k = X_j(t) + Q_j(t) + \Psi_j(z_1, \ldots, z_r, t) \qquad (j = 1, \ldots, r) \tag{1.10}$$

The functions $X_\mu(t)$, $Q_\mu(t)$, $\Psi_\mu(z_1, \ldots, z_r, t)$ for which $\mu \neq \sigma_l$ $(l = 1, \ldots, n)$ are identically equal to zero in Equations (1.10).

The system of scalar differential equations (1.10) can be replaced by the matrix differential equations

$$\dot{z} + a(t) z = X(t) + Q(t) + \Psi(z_1, \ldots, z_r, t) \tag{1.11}$$

where z, $a(t)$, $X(t)$, $Q(t)$ and $\Psi(z_1, \ldots, z_r, t)$ are the following matrices

$$\begin{gathered} z = \|z_j\|, \quad a(t) = \|a_{jk}(t)\|, \quad X(t) = \|X_j(t)\|, \quad Q(t) = \|Q_j(t)\| \\ \Psi(z_1, \ldots, z_r, t) = \|\Psi_j(z_1, \ldots, z_r, t)\| \end{gathered} \tag{1.12}$$

We denote by $z(t_0) = \|z_j(t_0)\|$ the matrix of the values of the sought functions $z_j(t)$ at the initial instant of time $t = t_0$. Denoting by $\theta(t)$ the fundamental matrix of the matrix differential equation

$$\dot{z} + a(t) z = 0 \tag{1.13}$$

we obtain the following nonlinear matrix differential equation from the nonlinear matrix differential equation (1.11)

$$z(t) = N(t, t_0)\, z(t_0) + \int_{t_0}^{t} N(t, \tau)\,[X(\tau) + Q(\tau)]\, d\tau + \\ + \int_{t_0}^{t} N(t, \tau)\, \Psi(z_1(\tau), \ldots, z_r(\tau), \tau)\, d\tau \tag{1.14}$$

where

$$N(t, \tau) = \theta(t)\, \theta^{-1}(\tau) \tag{1.15}$$

is the weight function of the matrix differential equation (1.13). The symbol $\theta^{-1}(t)$ in the expression (1.15) denotes the inverse matrix.

Since the functions $X_\mu(t)$, $Q_\mu(t)$, $\Psi_\mu(z_1, \ldots, z_r, t)$ are identically equal to zero for $\mu \neq \sigma_l$ $(l = 1, \ldots, n)$, the system of scalar integral equations equivalent to the matrix equation (1.14) has the following form

$$z_j(t) = \sum_{k=1}^{r} N_{jk}(t, t_0)\, z_k(t_0) + \int_{t_0}^{t} \sum_{i=1}^{n} N_{j\sigma_i}(t, \tau)\,[X_{\sigma_i}(\tau) + Q_{\sigma_i}(\tau)]\, d\tau + \\ + \int_{t_0}^{t} \sum_{i=1}^{n} N_{j\sigma_i}(t, \tau)\, \Psi_{\sigma_i}(z_1(\tau), \ldots, z_r(\tau), \tau)\, d\tau \qquad (j = 1, \ldots, r) \tag{1.16}$$

Substituting the expressions (1.7) for $X_{\sigma_i}(t)$, $Q_{\sigma_i}(t)$ and $\Psi_{\sigma_i}(z_1, \ldots, z_r, t)$ into Equations (1.16), we obtain

$$z_j(t) = \sum_{k=1}^{r} N_{jk}(t, t_0)\, z_k(t_0) + \int_{t_0}^{t} \sum_{l=1}^{n} \sum_{i=1}^{n} N_{j\sigma_i}(t, \tau)\, \frac{B_{li}(\tau)}{\Delta^*(\tau)}\,[x_l(\tau) + q_l(\tau) + \\ + \psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)]\, d\tau \qquad (j = 1, \ldots, r) \tag{1.17}$$

Denoting

$$W_{jl}(t, \tau) = \sum_{i=1}^{n} N_{j\sigma_i}(t, \tau)\, \frac{B_{li}(\tau)}{\Delta^*(\tau)} \qquad \binom{j = 1, \ldots, r}{l = 1, \ldots, n} \tag{1.18}$$

$$g_j(t) = \sum_{k=1}^{r} N_{jk}(t, t_0)\, z_k(t_0) + \sum_{l=1}^{n} \int_{t_0}^{t} W_{jl}(t, \tau)\, x_l(\tau)\, d\tau \qquad (j = 1, \ldots, r) \tag{1.19}$$

we write the system of integral equations (1.17) in the form

$$z_j(t) = g_j(t) + \sum_{l=1}^{n} \int_{t_0}^{t} W_{jl}(t, \tau)\, q_l(\tau)\, d\tau +$$

$$+\sum_{l=1}^{n}\int_{t_0}^{t} W_{jl}(t,\tau)\,\psi_l(z_1(\tau),\ldots,z_r(\tau),\tau)\,d\tau \qquad (j=1,\ldots,r) \tag{1.20}$$

We shall require that some of the phase coordinates of the system z_{p_ν} ($\nu = 1, \ldots, m$) assume given values r_{p_ν} at the time t_1. If the number m of the phase coordinates whose values at the instant t_1 are prescribed in advance is smaller than the number n of the possible additional forces, then we assume

$$q_{\alpha_1}(t) \equiv q_{\alpha_2}(t) \equiv \ldots \equiv q_{\alpha_{n-m}}(t) \equiv 0 \tag{1.21}$$

The additional forces $q_{s_1}(t)$, $q_{s_2}(t)$, ..., $q_{s_m}(t)$, which should be determined in such a way that the following conditions are satisfied

$$z_{p_\nu}(t_1) = r_{p_\nu} \qquad (\nu = 1,\ldots,m) \tag{1.22}$$

will be assumed as step functions, i.e. having constant values in the interval (t_0, t_1)

$$q_{s_i}(t) \equiv q_{s_i}(t_0) \qquad (t_0 \leqslant t < t_1) \qquad (i = 1,\ldots,m) \tag{1.23}$$

This choice of the functions $q_{s_i}(t)$ ($i = 1, \ldots, m$) is, generally speaking, possible because only the values of the phase coordinates z_{p_ν} ($\nu = 1, \ldots, m$) at the time $t = t_1$ are specified, and no limitations are imposed on the variations of the functions $z_{p_\nu}(t)$ ($\nu = 1, \ldots, m$) in the interval (t_0, t_1). We exclude from our discussion the cases in which any one or any group of the equations in the system (1.1) are independent of the remaining equations.

The functions $z_j(t)$ ($j = 1, \ldots, r$) sought in the interval (t_0, t_1) and the unknown quantities $q_{s_i}(t_0)$ ($i = 1, \ldots, m$) are determined by the following system of equations, according to (1.20), (1.22) and (1.23)

$$z_j(t) = g_j(t) + \sum_{i=1}^{m} F_{js_i}(t)\; q_{s_i}(t_0) +$$

$$+\sum_{l=1}^{n}\int_{t_0}^{t} W_{jl}(t,\tau)\,\psi_l(z_1(\tau),\ldots,z_r(\tau),\tau)\,d\tau \qquad \begin{pmatrix} t_0 \leqslant t \leqslant t_1 \\ j = 1,\ldots,r \end{pmatrix}$$

(1.24)

$$r_{p_\nu} - g_{p_\nu}(t_1) = \sum_{i=1}^{m} F_{p_\nu s_i}(t_1)\, q_{s_i}(t_0) +$$

$$+\sum_{l=1}^{n}\int_{t_0}^{t_1} W_{p_\nu l}(t_1, \tau)\,\psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\,d\tau \qquad (\nu = 1, \ldots, m)$$

Here, $F_{js_i}(t)$ denotes the known functions

$$F_{js_i}(t) = \int_{t_0}^{t} W_{js_i}(t, \tau)\,d\tau \qquad \begin{pmatrix} j = 1, \ldots, r \\ i = 1, \ldots, m \end{pmatrix} \tag{1.25}$$

Equations (1.24) can be transformed in the following way. The second group of Equations (1.24) implies

$$q_{s_i}(t_0) = \frac{1}{\Delta(t_1)} K_{s_i}(t_1) - \tag{1.26}$$

$$-\frac{1}{\Delta(t_1)}\sum_{\mu=1}^{m} A_{p_\mu s_i}(t_1)\int_{t_0}^{t_1}\sum_{l=1}^{n} W_{p_\mu l}(t_1, \tau)\,\psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\,d\tau$$

$$(i = 1, \ldots, m)$$

where

$$\Delta(t_1) = \begin{vmatrix} F_{p_1 s_1}(t_1) & F_{p_1 s_2}(t_1) \ldots & F_{p_1 s_m}(t_1) \\ \ldots & \ldots\ldots & \ldots\ldots \\ F_{p_m s_1}(t_1) & F_{p_m s_2}(t_1) \ldots & F_{p_m s_m}(t_1) \end{vmatrix} \tag{1.27}$$

$$K_{s_i}(t_1) = \sum_{\mu=1}^{m} A_{p_\mu s_i}(t_1)\,[r_{p_\mu} - g_{p_\mu}(t_1)] \qquad (i = 1, \ldots, m) \tag{1.28}$$

and $A_{p_\mu s_i}(t_1)$ (μ, $i = 1, \ldots, m$) are the algebraic complements of the elements $F_{p_\mu s_i}$ in the determinant (1.27). With the notations

$$k_{s_i}(t_1) = \frac{1}{\Delta(t_1)} K_{s_i}(t_1) \qquad (i = 1, \ldots, m) \tag{1.29}$$

$$U_{s_i l}(t_1, \tau) = \frac{1}{\Delta(t_1)}\sum_{\mu=1}^{m} A_{p_\mu s_i}(t_1)\,W_{p_\mu l}(t_1, \tau) \qquad \begin{pmatrix} i = 1, \ldots, m \\ l = 1, \ldots, n \end{pmatrix} \tag{1.30}$$

we reduce the expressions (1.26) to the form

$$q_{s_i}(t_0) = k_{s_i}(t_1) - \sum_{l=1}^{n}\int_{t_0}^{t_1} U_{s_i l}(t_1, \tau)\,\psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\,d\tau \qquad (i = 1, \ldots, m) \tag{1.31}$$

Substituting the expressions (1.31) for $q_{s_i}(t_0)$ into the first group of Equations (1.24), we obtain the following system of nonlinear integral equations

$$z_j(t) = G_j(t) - \sum_{i=1}^{m}\sum_{l=1}^{n} F_{js_i}(t)\int_{t_0}^{t_1} U_{s_il}(t_1, \tau)\,\psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\,d\tau +$$

$$+ \sum_{l=1}^{n}\int_{t_0}^{t} W_{jl}(t, \tau)\,\psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\,d\tau \qquad \begin{pmatrix} t_0 \leqslant t \leqslant t_1 \\ j = 1, \ldots, r \end{pmatrix} \tag{1.32}$$

where

$$G_j(t) = g_j(t) + \sum_{i=1}^{m} F_{js_i}(t)\,k_{s_i}(t_1) \qquad (j = 1, \ldots, r) \tag{1.33}$$

We note that the number of equations in the system (1.32) decreases if the nonlinear functions $\psi_l(z_1, \ldots, z_r, t)$ $(l = 1, \ldots, n)$ do not depend on some phase coordinates z_ρ of the system. For example, if the nonlinear functions ψ_l $(l = 1, \ldots, n)$ contain only one phase coordinate z_k

$$\psi_l = \psi_l(z_k(t), t) \qquad (l = 1, \ldots, n) \tag{1.34}$$

then, according to (1.32), we have to solve the following nonlinear integral equation with respect to the unknown function $z_k(t)$

$$z_k(t) = G_k(t) - \sum_{i=1}^{m}\sum_{l=1}^{n} F_{ks_i}(t)\int_{t_0}^{t_1} U_{s_il}(t_1, \tau)\,\psi_l(z_k(\tau), \tau)\,d\tau +$$

$$+ \sum_{l=1}^{n}\int_{t_0}^{t} W_{kl}(t, \tau)\,\psi_l(z_k(\tau), \tau)\,d\tau \qquad (t_0 \leqslant t \leqslant t_1) \tag{1.35}$$

The remaining phase coordinates z_ρ $(\rho = 1, \ldots, k-1, k+1, \ldots, r)$ will be expressed in terms of the integrals

$$z_\rho(t) = G_\rho(t) - \sum_{i=1}^{m}\sum_{l=1}^{n} F_{\rho s_i}(t)\int_{t_0}^{t_1} U_{s_il}(t_1, \tau)\,\psi_l(z_k(\tau), \tau)\,d\tau +$$

$$+ \sum_{l=1}^{n}\int_{t_0}^{t} W_{\rho l}(t, \tau)\,\psi_l(z_k(\tau), \tau)\,d\tau \qquad (t_0 \leqslant t \leqslant t_1) \tag{1.36}$$

In this, the additional forces being investigated

$$q_{s_i}(t) \qquad (i = 1, \ldots, m)$$

will be according to (1.23) and (1.31)

$$q_{s_i}(t) = q_{s_i}(t_0) = k_{s_i}(t_1) - \sum_{l=1}^{n}\int_{t_0}^{t_1} U_{s_il}(t_1, \tau)\,\psi_l(z_k(\tau), \tau)\,d\tau \qquad \begin{pmatrix} t_0 \leqslant t < t_1 \\ i = 1, \ldots, m \end{pmatrix} \tag{1.37}$$

2. We shall consider now the case in which the number of additional forces being realized in the controlled system is smaller than the number of the phase coordinates which should assume prescribed values at a given instant of time $t = t_1$.

For definiteness, let us consider only one additional external force which has to be used in such a way that the conditions (1.22) be satisfied, i.e.

$$z_{p_\nu}(t_1) = r_{p_\nu} \qquad (\nu = 1, \ldots, m)$$

In order to solve this problem we divide the time interval (t_0, t_1) into m identical or different subintervals (t_0, T_1), (T_1, T_2), ..., (T_{m-1}, t_1).

We assume the function $q_s(t)$ of the step type, and we denote by

$$q_s(T_0),\ q_s(T_1), \ldots, q_s(T_{m-1})$$

its values in the respective subintervals.

Equations (1.20), determining the motion of the system, assume the form

$$z_j(t) = g_j(t) + \sum_{i=0}^{m-1} q_s(T_i)\,1\,(t - T_i)\int_{T_i}^{\sigma_i} W_{js}(t, \tau)\,d\tau +$$

$$+ \sum_{l=1}^{n}\int_{t_0}^{t} W_{jl}(t, \tau)\,\psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\,d\tau \qquad \begin{pmatrix} t_0 \leqslant t \leqslant t_1 \\ j = 1, \ldots, r \end{pmatrix} \tag{2.1}$$

where

$$T_0 = t_0, \qquad T_m = t_1 \tag{2.2}$$

$$\sigma_i = t + (T_{i+1} - t)\,1\,(t - T_{i+1}) \qquad (i = 0, 1, \ldots, m-1) \tag{2.3}$$

$$1\,(\xi) = \begin{cases} 0 & (\xi < 0) \\ 1 & (\xi \geqslant 0) \end{cases} \tag{2.4}$$

The conditions (1.22) reduce now to the form

$$r_{p_\nu} - g_{p_\nu}(t_1) = \sum_{i=0}^{m-1} V_{p_\nu s}(T_i)\,q_s(T_i) +$$

$$+ \sum_{l=1}^{n}\int_{t_0}^{t_1} W_{p_\nu l}(t_1, \tau)\,\psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\,d\tau \qquad (\nu = 1, \ldots, m) \tag{2.5}$$

where

$$V_{p_\nu s}(T_i) = \int\limits_{T_i}^{T_{i+1}} W_{p_\nu s}(t_1, \tau)\, d\tau \qquad \begin{pmatrix} \nu = 1, \ldots, m \\ i = 0, 1, \ldots, m-1 \end{pmatrix} \tag{2.6}$$

It follows from Equations (2.5) that

$$q_s(T_i) = \varkappa_i(t_1) - \sum_{l=1}^{n} \int\limits_{t_0}^{t_1} \Xi_{il}(t_1, \tau)\, \psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\, d\tau \qquad (i = 0, 1, \ldots, m-1) \tag{2.7}$$

where

$$\varkappa_i(t_1) = \frac{1}{\Lambda} \sum_{\mu=1}^{m} C_{p_\mu i} [r_{p_\mu} - g_{p_\mu}(t_1)] \qquad (i = 0, 1, \ldots, m-1) \tag{2.8}$$

$$\Xi_{il}(t_1, \tau) = \frac{1}{\Lambda} \sum_{\mu=1}^{m} C_{p_\mu i} W_{p_\mu l}(t_1, \tau) \qquad \begin{pmatrix} i = 0, 1, \ldots, m-1 \\ l = 1, \ldots, n \end{pmatrix} \tag{2.9}$$

$$\Lambda = \begin{vmatrix} V_{p_1 s}(T_0) & V_{p_1 s}(T_1) & \ldots & V_{p_1 s}(T_{m-1}) \\ \cdots & \cdots & \cdots & \cdots \\ V_{p_m s}(T_0) & V_{p_m s}(T_1) & \ldots & V_{p_m s}(T_{m-1}) \end{vmatrix} \tag{2.10}$$

and $C_{p_\mu i}$ ($\mu = 1, \ldots, m$; $i = 0, 1, \ldots, m-1$) denote the algebraic complements of the elements $V_{p_\mu s}(T_i)$ in the determinant (2.10).

Substituting the expressions (2.7) for $q_s(T_i)$ into Equations (2.1), we obtain the following system of nonlinear differential equations which determine the variations of the phase coordinates

$$z_j(t) = \Gamma_j(t) - \sum_{i=0}^{m-1} \sum_{l=1}^{n} \chi_{ji}(t) \int\limits_{t_0}^{t_1} \Xi_{il}(t_1, \tau)\, \psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\, d\tau +$$

$$+ \sum_{l=1}^{n} \int\limits_{t_0}^{t} W_{jl}(t, \tau)\, \psi_l(z_1(\tau), \ldots, z_r(\tau), \tau)\, d\tau \qquad \begin{pmatrix} t_0 \leqslant t \leqslant t_1 \\ j = 1, \ldots, r \end{pmatrix} \tag{2.11}$$

where

$$\Gamma_j(t) = g_j(t) + \sum_{i=0}^{m-1} \varkappa_i(t_1)\, \chi_{ji}(t) \qquad (j = 1, \ldots, r) \tag{2.12}$$

$$\chi_{ji}(t) = 1(t - T_i) \int\limits_{T_i}^{\sigma_i} W_{js}(t, \tau)\, d\tau \qquad \begin{pmatrix} j = 1, \ldots, r \\ i = 0, 1, \ldots, m-1 \end{pmatrix} \tag{2.13}$$

If the functions ψ_l ($l = 1, \ldots, n$) do not depend on some of the phase coordinates of the system z_ρ, the number of nonlinear integral equations in the system (2.11) becomes smaller. If, for example, the functions ψ_l have the form (1.34)

$$\psi_l = \psi_l(z_k(t), t) \qquad (l = 1, \ldots, n)$$

then, according to (2.11), we have the following nonlinear integral equation with respect to the unknown function $z_k(t)$

$$z_k(t) = \Gamma_k(t) - \sum_{i=0}^{m-1} \sum_{l=1}^{n} \chi_{ki}(t) \int_{t_0}^{t_1} \Xi_{il}(t_1, \tau)\, \psi_l(z_k(\tau), \tau)\, d\tau + \\ + \sum_{l=1}^{n} \int_{t_0}^{t} W_{kl}(t, \tau)\, \psi_l(z_k(\tau), \tau)\, d\tau \qquad (t_0 \leqslant t \leqslant t_1) \tag{2.14}$$

The remaining phase coordinates can be expressed in terms of the integrals

$$z_\rho(t) = \Gamma_\rho(t) - \sum_{i=0}^{m-1} \sum_{l=1}^{n} \chi_{\rho i}(t) \int_{t_0}^{t_1} \Xi_{il}(t_1, \tau)\, \psi_l(z_k(\tau), \tau)\, d\tau + \\ + \sum_{l=1}^{n} \int_{t_0}^{t} W_{\rho l}(t, \tau)\, \psi_l(z_k(\tau), \tau)\, d\tau \qquad \begin{pmatrix} t_0 \leqslant t \leqslant t_1 \\ \rho = 1, \ldots, k-1, k+1, \ldots r \end{pmatrix} \tag{2.15}$$

According to (2.7) the values of the additional external force $q_s(t)$, which is a step function in the time intervals (T_i, T_{i+1}) $(i = 0, 1, \ldots, m-1)$ are the following

$$q_s(T_i) = \varkappa_i(t_1) - \sum_{l=1}^{n} \int_{t_0}^{t_1} \Xi_{il}(t_1, \tau)\, \psi_l(z_k(\tau), \tau)\, d\tau \qquad (i = 0, 1, \ldots, m-1) \tag{2.16}$$

The method presented allows for realization of a given state of motion in the m-dimensional phase space $(z_{p_1}, \ldots, z_{p_m})$. In this, if the number of the additional forces $q_{s_i}(t)$ is smaller than m, the conditions of the type (1.22) are satisfied at the discrete points $t_1, t_2, \ldots$

For the solution of the integral equations of the type (1.32) or (2.11), which determine the additional external forces $q_{s_i}(t)$, it is necessary to apply numerical methods [3,4,5].

3. In the particular case in which only one phase coordinate z_p has a prescribed value and the equations of motion contain only one nonlinear function

$$\psi_\lambda = \psi_\lambda(z_k(t), t) \tag{3.1}$$

the additional external force should be selected in such a way that the condition

$$z_p(t_1) = r_p \tag{3.2}$$

be satisfied.

In this case Equations (1.24) assume the form

$$\begin{gathered} z_j(t) = g_j(t) + F_{js}(t)\, q_s(t_0) + \int_{t_0}^{t} W_{j\lambda}(t,\tau)\, \psi_\lambda(z_k(\tau),\tau)\, d\tau \qquad (j = 1,\ldots,r) \\ r_p - g_p(t_1) = F_{ps}(t_1)\, q_s(t_0) + \int_{t_0}^{t_1} W_{p\lambda}(t_1,\tau)\, \psi_\lambda(z_k(\tau),\tau)\, d\tau \end{gathered} \tag{3.3}$$

where, according to (1.25)

$$F_{js}(t) = \int_{t_0}^{t} W_{js}(t,\tau)\, d\tau \qquad (j = 1,\ldots,r) \tag{3.4}$$

From the last of Equations (3.3) it follows that

$$q_s(t_0) = \frac{1}{F_{ps}(t_1)} \left[r_p - g_p(t_1) - \int_{t_0}^{t_1} W_{p\lambda}(t_1,\tau)\, \psi_\lambda(z_k(\tau),\tau)\, d\tau \right] \tag{3.5}$$

while the first group of Equations (3.3) assumes the form

$$\begin{gathered} z_j(t) = g_j(t) + \frac{F_{js}(t)}{F_{ps}(t_1)} \left[r_p - g_p(t_1) - \int_{t_0}^{t_1} W_{p\lambda}(t_1,\tau) \psi_\lambda(z_k(\tau),\tau) d\tau \right] + \\ + \int_{t_0}^{t} W_{j\lambda}(t;\tau)\, \psi_\lambda(z_k(\tau),\tau)\, d\tau \qquad (j = 1,\ldots r) \end{gathered} \tag{3.6}$$

According to (3.6) we have the following nonlinear integral equation for the unknown function $z_k(t)$

$$\begin{gathered} z_k(t) = g_k(t) + \frac{F_{ks}(t)}{F_{ps}(t_1)} \left[r_p - g_p(t_1) - \int_{t_0}^{t_1} W_{p\lambda}(t_1,\tau)\, \psi_\lambda(z_k(\tau),\tau)\, d\tau \right] + \\ + \int_{t_0}^{t} W_{k\lambda}(t,\tau)\, \psi_\lambda(z_k(\tau),\tau)\, d\tau \qquad (t_0 \leqslant t \leqslant t_1) \end{gathered} \tag{3.7}$$

The remaining phase coordinates $z_\rho (\rho = 1, \ldots, k-1, k+1, \ldots, r)$ can be expressed in terms of the integrals

$$z_\rho(t) = g_\rho(t) + \frac{F_{\rho s}(t)}{F_{ps}(t_1)}\left[r_p - g_p(t_1) - \int_{t_0}^{t_1} W_{p\lambda}(t_1, \tau)\,\psi_\lambda(z_k(\tau), \tau)\,d\tau\right] +$$

$$+ \int_{t_0}^{t} W_{\rho\lambda}(t, \tau)\,\psi_\lambda(z_k(\tau), \tau)\,d\tau \qquad (t_0 \leqslant t \leqslant t_1) \tag{3.8}$$

The additional external force $q_s(t)$, according to (1.23) and (3.5), is

$$q_s(t) \equiv q_s(t_0) = k_s(t_1) - \frac{1}{F_{ps}(t_1)}\int_{t_0}^{t_1} W_{p\lambda}(t_1, \tau)\,\psi_\lambda(z_k(\tau), \tau)\,d\tau \qquad (t_0 \leqslant t < t_1) \tag{3.9}$$

where

$$k_s(t_1) = \frac{1}{F_{ps}(t_1)}\left[r_p - g_p(t_1)\right] \tag{3.10}$$

4. We shall consider now an application of the method to the problem of accelerated north-seeking action of a gyrocompass with a nonlinear restoring force.

The equations of precessional motion of a gyrocompass can be written in the form

$$\begin{gathered} -H\dot{\alpha} + lP\beta + lP(1-\rho)\vartheta = HU\sin\varphi + Q\ (t) \\ H\dot{\beta} + HU\cos\varphi\cdot\alpha + M\alpha^3 = 0, \quad \dot{\vartheta} + F\vartheta + F\beta = 0 \end{gathered} \tag{4.1}$$

Here, α is the azimuth angle of the spin axis, β is the spin-axis tilt, ϑ is the angle of the fluid surface of the hydraulic damper above the spin axis, H is the angular momentum of the rotor, lP is the gyro pendulous factor, U is the earth's angular velocity, φ is the latitude of the point of observation.

Equations (4.1) contain the nonlinear restoring force $-M\alpha^3$ which represents a moment with respect to the vertical axis of the gyrocompass and which accelerates the coincidence of the gyrocompass and the meridional plane. This restoring force is effective at larger deviations of the gyrocompass as it decreases the period of the natural vibrations of the gyrocompass for large values of α.

Here, $Q(t)$ denotes the additional external force representing the moment with respect to the east-west axis of the gyrocompass, imposed in order to bring the spin axis into the meridional plane at a given instant of time. The law of variation of this force in time will be determined in the following.

With the notations

$$z_1 = \alpha, \qquad z_2 = \beta - \frac{HU \sin \varphi}{\rho l P}, \qquad z_3 = \vartheta + \frac{HU \sin \varphi}{\rho l P}, \qquad k^2 = \frac{lPU \cos \varphi}{H}$$

$$q_1(t) = -\frac{Q(t)}{H}, \qquad \psi_2(z_1) = -\zeta z_1^3, \qquad \zeta = \frac{M}{H} \tag{4.2}$$

Equations (4.1) can be reduced to the following form

$$\dot{z}_1 - \frac{k^2}{U \cos \varphi} z_2 - \frac{k^2 (1-\rho)}{U \cos \varphi} z_3 = q_1(t)$$

$$\dot{z}_2 + U \cos \varphi \, z_1 = \psi_2(z_1), \qquad \dot{z}_3 + F z_2 + F z_3 = 0 \tag{4.3}$$

The system of scalar equations (4.3) can be replaced by the matrix equation

$$\dot{z} + az = q(t) + \psi(z_1, z_2, z_3) \tag{4.4}$$

where

$$z = \begin{Vmatrix} z_1 \\ z_2 \\ z_3 \end{Vmatrix}, \qquad a = \begin{Vmatrix} 0 & -\dfrac{k^2}{U \cos \varphi} & -\dfrac{k^2 (1-\rho)}{U \cos \varphi} \\ U \cos \varphi & 0 & 0 \\ 0 & F & F \end{Vmatrix}$$

$$q(t) = \begin{Vmatrix} q_1(t) \\ 0 \\ 0 \end{Vmatrix}, \qquad \psi(z_1, z_2, z_3) = \begin{Vmatrix} 0 \\ \psi_2(z_1) \\ 0 \end{Vmatrix} \tag{4.5}$$

We denote by $N(t) = \| N_{jk}(t) \|$ the weight function of the matrix differential equation

$$\dot{z} + az = 0 \tag{4.6}$$

Since the elements of the matrix a are constant quantities, the function $N(t)$ is determined by the operational relation

$$\frac{p\Phi(p)}{\Delta(p)} \doteqdot N(t) \tag{4.7}$$

Here, $\Phi(p)$ denotes the adjoint matrix of the matrix $e(p)$, $\Delta(p)$ is the determinant of the matrix $e(p)$, while the matrix $e(p)$ has the form

$$e(p) = Ep + a \tag{4.8}$$

where E is the unit matrix.

The following matrix integral equation is equivalent to Equation (4.4) with the initial conditions

$$z(t) = N(t)\,z(0) + \int_0^t N(t-\tau)\,q(\tau)\,d\tau + \int_0^t N(t-\tau)\,\psi(z_1(\tau)),\, z_2(\tau),\quad z_3(\tau))\,d\tau \tag{4.9}$$

The matrix integral equation (4.9) can be replaced by a system of scalar integral equations which, according to (4.5), have the form

$$z_j(t) = \sum_{k=1}^{3} N_{jk}(t)\,z_k(0) + \int_0^t N_{j1}(t\,\tau)\ q_1\ (\tau)\,d\,\tau + \int_0^t N_{j2}(t-\tau)\,\psi_2(z_1(\tau))\,d\tau$$
$$(j = 1, 2, 3) \tag{4.10}$$

We require now that at the time $t = t_1$ the gyrocompass indicate meridian, i.e. the conditions be satisfied:

$$z_j(t_1) = 0 \qquad (j = 1, 2, 3) \tag{4.11}$$

According to (4.10), conditions (4.11) assume the form

$$\int_0^{t_1} N_{j1}(t_1-\tau)q_1(\tau)d\tau = -\sum_{k=1}^{3} N_{jk}(t_1)\,z_k(0) - \int_0^{t_1} N_{j2}(t_1-\tau)\,\psi_2(z_1(\tau))\,d\tau$$
$$(j = 1, 2, 3) \tag{4.12}$$

The time interval $(0,\ t_1)$ will be divided into three equal subintervals $(0,\ T_1)$, $(T_1,\ T_2)$, $(T_2,\ t_1)$. We assume the function $q_1(t)$ of a step type, and we denote by $q_1(T_0)$, $q_1(T_1)$, $q_1(T_2)$ its values in each of these subintervals. We note that, according to (2.2), it is

$$T_0 = 0, \qquad T_3 = t_1 \tag{4.13}$$

The integral equations (4.10), determining the motion of the system, are now of the form

$$z_j(t) = g_j(t) + \sum_{i=0}^{2} q_1(T_i)\,1\,(t-T_i)\int_{T_i}^{\sigma_i} N_{j1}(t-\tau)\,d\tau + \int_0^t N_{j2}(t-\tau)\,\psi_2(z_1(\tau))\,d\tau$$
$$(0 \leqslant t \leqslant t_1;\ j = 1, 2, 3) \tag{4.14}$$

Here

$$g_j(t) = \sum_{k=1}^{3} N_{jk}(t)\,z_k(0) \qquad (j = 1, 2, 3) \tag{4.15}$$

The quantities σ_i, according to (2.3), are determined by the formulas

$$\sigma_i = t + (T_{i+1} - t)\,1\,(t - T_{i+1}) \qquad (i = 0, 1, 2) \tag{4.16}$$

and $1(\xi)$ denotes, as in (2.4) above, the unit step function.

The conditions (4.12) can be reduced now to the following form

$$\sum_{i=0}^{2} V_{j1}(T_i)\, q_1(T_i) = -g_j(t_1) - \int_0^{t_1} N_{j2}(t_1-\tau)\, \psi_2(z_1(\tau))\, d\tau \qquad (j=1,2,3) \tag{4.17}$$

where

$$V_{j1}(T_i) = \int_{T_i}^{T_{i+1}} N_{j1}(t_1-\tau)\, d\tau \qquad (j=1,2,3;\ i=0,1,2) \tag{4.18}$$

It follows from Equations (4.17) that

$$q_1(T_i) = \varkappa_i(t_1) - \int_0^{t_1} \Xi_{i2}(t_1-\tau)\, \psi_2(z_1(\tau))\, d\tau \qquad (i=0,1,2) \tag{4.19}$$

where

$$\varkappa_i(t_1) = -\frac{1}{\Lambda} \sum_{\mu=1}^{3} C_{\mu i} g_\mu(t_1) \qquad (i=0,1,2) \tag{4.20}$$

$$\Xi_{i2}(t_1-\tau) = \frac{1}{\Lambda} \sum_{\mu=1}^{3} C_{\mu i} N_{\mu 2}(t_1-\tau) \qquad (i=0,1,2) \tag{4.21}$$

$$\Lambda = \begin{vmatrix} V_{11}(T_0) & V_{11}(T_1) & V_{11}(T_2) \\ V_{21}(T_0) & V_{21}(T_1) & V_{21}(T_2) \\ V_{31}(T_0) & V_{31}(T_1) & V_{31}(T_2) \end{vmatrix} \tag{4.22}$$

and $C_{\mu i}$ (μ = 1, 2, 3; i = 0, 1, 2) denote the algebraic complements of the elements $V_{\mu 1}(T_i)$ in the determinant (4.22).

Substituting the expressions (4.12) for $q_1(T_i)$ (i = 0, 1, 2) into (4.14), we obtain the following integral equations with respect to the unknown function $z_1(t)$

$$z_1(t) = \Gamma_1(t) - \sum_{i=0}^{2} \chi_{1i}(t) \int_0^{t_1} \Xi_{i2}(t_1-\tau)\, \psi_2(z_1(\tau))\, d\tau + \\ + \int_0^{t} N_{12}(t-\tau)\, \psi_2(z_1(\tau))\, d\tau \qquad (0 \leqslant t \leqslant t_1) \tag{4.23}$$

The generalized coordinates $z_2(t)$ and $z_3(t)$ can be expressed as the integrals

$$z_v(t) = \Gamma_v(t) - \sum_{i=0}^{2} \chi_{vi}(t) \int_0^{t_1} \Xi_{i2}(t_1 - \tau) \psi_2(z_1(\tau))\, d\tau +$$

$$+ \int_0^t N_{v2}(t - \tau) \psi_2(z_1(\tau))\, d\tau \qquad (0 \leqslant t \leqslant t_1; \quad v = 2, 3) \tag{4.24}$$

Here

$$\Gamma_j(t) = g_j(t) + \sum_{i=0}^{2} \varkappa_i(t_1) \chi_{ji}(t) \qquad (j = 1, 2, 3) \tag{4.25}$$

$$\chi_{ji}(t) = 1(t - T_i) \int_{T_i}^{\sigma_i} N_{j1}(t - \tau)\, d\tau \qquad (j = 1, 2, 3;\ i = 0, 1, 2) \tag{4.26}$$

Having determined the function $z_1(t)$ from the integral equation (4.23), in the time interval $0 < t < t_1$, we can calculate the values $q_1(T_i)$ (i = 0, 1, 2) from the Formulas (4.19). In this way we obtain the time dependence of the additional external force $q_1(t)$ which assures the coincidence of the gyrocompass and the meridional plane at the time $t = t_1$.

The integral equation (4.23) has been solved with the use of an electronic computer for the following values of the parameters (the author uses this opportunity to express his gratitude to V.A. Cherpasov for programming the computation)

$$k^2 = 1.53921 \times 10^{-6}\ \mathrm{sec}^{-2}, \quad \rho = 0.38, \quad F = 1.5 \times 10^{-3}\ \mathrm{sec}^{-1}$$

$$U \cos \varphi = 4.11368 \times 10^{-5}\ \mathrm{sec}^{-1}, \quad \zeta = 0.4 \times 10^{-3}\ \mathrm{sec}^{-1}$$

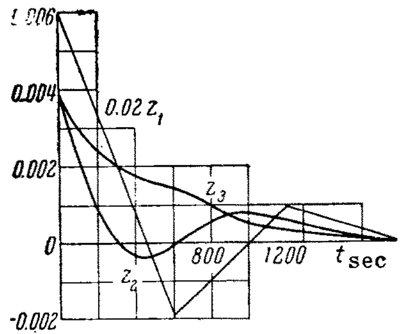
t sec	z_1	$10^3 z_2$	$10^3 z_3$	t sec	z_1	$10^3 z_2$	$10^3 z_3$
0	0.30000	4.0000	4.0000	950	—0,01466	0,7333	0.6473
50	0.27385	2.9349	3.4620	1000	—0.00262	0.7511	0,5467
100	0.24531	2.0490	3.0332	1050	0.00917	0.7444	0.4530
150	0.21483	1.3304	2.6933	1100	0.02089	0.7134	0,3675
200	0.18280	0.7628	2.4241	1150	0.03244	0.6581	0.2913
250	0.14956	0.3279	2.2105	1200	0.04378	0.5786	0.2254
300	0.11539	0.0080	2.0394	1250	0.04093	0.4899	0.1706
350	0.08054	—0.2130	1.9001	1300	0.03786	0.4076	0.1259
400	0.04521	—0.3478	1.7837	1350	0.03459	0.3321	0.0902
450	0.00957	—0.4048	1.6824	1400	0.03116	0.2638	0.0622
500	—0.02621	—0.3875	1.5899	1450	0.02758	0.2028	0.0409
550	—0.06200	—0.2949	1.5001	1500	0.02386	0.1496	0.0253
600	—0.09764	—0.1200	1.4072	1550	0.02004	0.1042	0.0144
650	—0.08621	0.0848	1.3065	1600	0.01614	0.0668	0.0072
700	—0.07455	0,2606	1.1994	1650	0.01216	0.0377	0.0030
750	—0.06272	0.4083	1.0883	1700	0.00813	0.0168	0.0009
800	—0.05076	0.5287	0.9756	1750	0.00407	0.0042	0.0001
850	—0.03873	0.6226	0.8633	1800	0.00000	0.0000	0.0000
900	—0.02668	0.6906	0.7533				

The time interval during which the gyrocompass should be brought into meridional plane is $t_1 = 1800$ sec (see the figure). The initial deviations are

$$z_1(0) = 0.3, \quad z_2(0) = 0.004, \quad z_3(0) = 0.004$$

Equations (4.23) have been solved by the method of successive approximations. As the zero approximation

$$z_1^{(0)}(t) = \Gamma_1(t).$$



has been assumed. The subsequent approximations $z_1^{(n)}(t)$ $(n = 1, 2, \ldots)$ have been determined from the expressions

$$z_1^{(n)}(t) = \Gamma_1(t) - \sum_{i=0}^{2} \chi_{1i}(t) \int_0^{t_1} \Xi_{i2}(t_1 - \tau)\, \psi_2(z_1^{(n-1)}(\tau))\, d\tau + \int_0^t N_{12}(t-\tau)\, \psi_2(z_1^{(n-1)}(\tau))\, d\tau \qquad (0 \leqslant t \leqslant t_1) \tag{4.27}$$

The following values have been obtained for $q_1(T_0)$, $q_1(T_1)$, $q_1(T_2)$

$$q_1(T_0) = -0.73856 \cdot 10^{-3}\ \mathrm{sec}^{-1}$$
$$q_1(T_1) = 0.19759 \cdot 10^{-3}\ \mathrm{sec}^{-1}$$
$$q_1(T_2) = -0.08157 \cdot 10^{-3}\ \mathrm{sec}^{-1}$$

The north-seeking process of the gyrocompass is represented by the table of the functions $z_1(t)$, $z_2(t)$, $z_3(t)$ and the diagrams of these functions shown in the figure.

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